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WORD LENGTH IN SURFACE GROUPS WITH CHARACTERISTIC GENERATING SETS

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Abstract. A subset of a group is characteristic if it is invariant under every automorphism of the group. We study word length in fundamental groups of closed hyperbolic surfaces with respect to characteristic generating sets consisting of a finite union of orbits of the automorphism group, and show that the translation length of any element with nonzero crossing number is positive, and bounded below by a constant depending only (and explicitly) on a bound on the crossing numbers of generating elements. This answers a question of Benson Farb.

1. Introduction

Let S be a closed, orientable surface of genus at least 2. Then S admits a (nonunique) hyperbolic structure. Let $a \in \pi_1(S)$ and let $[a]$ denote the conjugacy class of a. Once we have fixed a hyperbolic structure on S, each nontrivial conjugacy class $[a]$ determines a unique closed (unparameterized) geodesic $\gamma(a)$ in the corresponding free homotopy class on S.

Note that $\gamma(a)$ only depends on the conjugacy class of a, so we could write $\gamma([a])$, but we choose the notation $\gamma(a)$ for simplicity.

Definition 1.1. If a is primitive, a *self-intersection* of $\gamma(a)$ is an unordered pair of geodesics $\tilde{\gamma}_1, \tilde{\gamma}_2$ in the universal cover of S which cover $\gamma(a)$ and have linked endpoints in the circle at infinity, up to the action of the deck group $\pi_1(S)$ on such pairs.

The *crossing number* of a, denoted $\mathrm{cr}(a)$, is the number of self-intersections of $\gamma(a)$. If $a = b^n$ then set $\mathrm{cr}(a) = n^2\mathrm{cr}(b)$.

Since $\gamma(a)$ only depends on the conjugacy class of a, it follows that cr also depends only on the conjugacy class of a.

Moreover, since linking data does not depend on the choice of hyperbolic structure on S (although the configuration of $\gamma(a)$ in S typically will), it follows that $\mathrm{cr}(a)$ does not depend on the choice of hyperbolic structure on S.

Lemma 1.2. *Crossing number is constant on orbits of* $\mathrm{Out}(\pi_1(S))$.

Proof. Choose a hyperbolic structure g on S, and let $[\phi]$ be an element of the mapping class group $\mathrm{MCG}(S)$, which is equal to $\mathrm{Out}(\pi_1(S))$ by Dehn-Nielsen, represented by a diffeomorphism $\phi\colon S \to S$. If $[b] = [\phi]([a])$ then $\gamma(b)$ in the $(\phi^{-1})^*g$ metric is the image of $\gamma(a)$ in the g metric under the diffeomorphism ϕ. $\square$

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For each non-negative integer n, let S_n be the subset consisting of all elements with crossing number at most n, and let S'_n be the subset of S_n consisting of primitive elements.

For each $a \in \pi_1(S)$ define $w_n(a)$ to be the word length of a with respect to the generating set S_n and $w'_n(a)$ to be the word length of a with respect to the generating set S'_n. Obviously $w'_n(a) \geq w_n(a)$ for any n, since $S'_n \subset S_n$.

Remark 1.3. There is an equality $\text{cr}(b) = \text{cr}(b^{-1})$ for every $b \in \pi_1(S)$, so the S_n and S'_n are symmetric generating sets.

Remark 1.4. Each S'_n is a finite union of primitive $\text{Aut}(\pi_1(S))$ orbits. Conversely, every finite union of primitive $\text{Aut}(\pi_1(S))$ orbits is contained in S'_n for some n, and every finite union of (not necessarily primitive) $\text{Aut}(\pi_1(S))$ orbits is contained in S_n for some n.

Benson Farb asked whether the diameter of $\pi_1(S)$ with respect to S_0 is infinite, and in general to understand the geometry of the Cayley graph of $\pi_1(S)$ with respect to $\text{Aut}(\pi_1(S))$-invariant generating sets. The purpose of this note is to prove the following theorem:

Theorem A. *There are constants $C_1(S), C_2(S), C_3(S)$ such that for any non-negative integers n, m and any $a \in \pi_1(S)$ with $cr(a) > 0$ there is an inequality*

$$w_n(a^m) \geq \frac{C_1 m}{\sqrt{n} + C_2} - C_3$$

Notice that the constants C_1, C_2, C_3 do *not* depend on n or on the choice of $a \in \pi_1(S)$.

Remark 1.5. The dependence of the constants C_i on the surface S is also probably unnecessary.

Question 1.6. *Let $a \in \pi_1(S)$ have $cr(a) = 0$. What is the growth rate of the word length of a^m with respect to the generating set S'_n?*

2. Counting quasimorphisms

2.1. **Quasimorphisms.** The usual way to obtain lower bounds on word length is to find a suitable function which is bounded on a generating set, which grows linearly on powers, and which is almost subadditive under multiplication. A rich source of such functions is *quasimorphisms*.

Definition 2.1. Let G be a group. A quasimorphism is a function $\phi : G \to \mathbb{R}$ for which there is a smallest real number $D(\phi) \geq 0$ called the *defect*, with the property that for all $a, b \in G$ there is an inequality

$$|\phi(a) + \phi(b) - \phi(ab)| \leq D(\phi)$$

A quasimorphism is *homogeneous* if $\phi(a^n) = n\phi(a)$ for all integers n.

If ϕ is a quasimorphism, the function

$$\overline{\phi}(a) := \lim_{n \to \infty} \frac{\phi(a^n)}{n}$$

is a homogeneous quasimorphism, whose defect satisfies $D(\overline{\phi}) \leq 2D(\phi)$. Homogeneous quasimorphisms are class functions. See e.g. [1] or [4] for details.

2.2. **Hyperbolic groups.** We assume the reader is familiar with the basic properties of hyperbolic geometry, hyperbolic groups and δ-hyperbolic spaces, quasigeodesics, Morse Lemma, convexity of distance function, etc. For a reference, see Bridson-Haefliger [2] or Gromov [7].

Let G be a group which is δ-hyperbolic with respect to a generating set A. Epstein-Fujiwara [6], generalizing a construction due to Brooks [3], define so-called *counting quasimorphisms* as follows.

Definition 2.2. Let σ be an oriented simplicial path in the Cayley graph $C_A(G)$, and let σ^{-1} denote the same path with the opposite orientation. A *copy* of σ is a translate $a \cdot \sigma$ with $a \in G$. For α an oriented simplicial path in $C_A(G)$, let $|\alpha|_\sigma$ denote the maximal number of disjoint copies of σ contained in α. For $a \in G$, define

$$c_\sigma(a) = d(\mathrm{id}, a) - \inf_\alpha(\mathrm{length}(\alpha) - |\alpha|_\sigma)$$

where the infimum is taken over all directed paths α in $C_A(G)$ from id to a.

Define a *counting quasimorphism* to be a function of the form

$$h_\sigma(a) := c_\sigma(a) - c_{\sigma^{-1}}(a)$$

In the sequel we always assume that the length of σ is at least 2. It is clear from this definition that the homogenization $\overline{h}$ of h_σ is a class function. A path α as above which achieves the infimum, for a given σ and $a \in G$ is called a *realizing path* for c_σ. Since the length of any path is a non-negative integer, a realizing path must exist for any a and any σ.

Realizing paths have the following universal property:

Lemma 2.3 (Epstein-Fujiwara, Prop. 2.2 [6])**.** *Any realizing path for c_σ is a (K, ϵ)-quasigeodesic, where*

$$K = \frac{\mathrm{length}(\sigma)}{\mathrm{length}(\sigma) - 1}, \quad \epsilon = \frac{2 \cdot \mathrm{length}(\sigma)}{\mathrm{length}(\sigma) - 1}$$

Notice by our hypothesis that $\mathrm{length}(\sigma)$ is at least 2 that $K \leq 2$ and $\epsilon \leq 4$. By the Morse Lemma there is a constant $C(\delta)$ such that every realizing path for c_σ must be contained in the C-neighborhood of any geodesic from id to a. In particular, one obtains the following lemma:

Lemma 2.4. *There is a constant $C(\delta)$ such that for any path $\sigma \in C_S(G)$ of length at least 2, and for any $a \in G$, if the C-neighborhood of any geodesic from id to a does not contain a copy of σ, then $c_\sigma(a) = 0$.*

Finally, the defect of h_σ (and therefore of $\overline{h}_\sigma$) can be controlled independently of σ:

Lemma 2.5 (Epstein-Fujiwara, Prop. 2.13 [6])**.** *Let σ be a path in $C_S(G)$ of length at least 2. Then there is a constant $C(\delta)$ such that $D(h_\sigma) \leq C$.*

3. Stability of crossings

If two geodesics in the hyperbolic plane intersect with a small angle, then as the geodesics are moved around slightly, the point of intersection might move wildly. Nevertheless, two geodesics which cross are close only in a compact region; for a small perturbation, the point of intersection cannot move outside that compact region.

Let α, β be two complete geodesics in the hyperbolic plane which cross transversely at p. Fix a small $\epsilon > 0$. Let α_ϵ denote the subset of α which intersects the 2ϵ-neighborhood of β, and similarly define β_ϵ. Let α', β' be two geodesics which contain segments $\alpha'_\epsilon, \beta'_\epsilon$ which are ϵ-close to $\alpha_\epsilon, \beta_\epsilon$ respectively. Then α', β' have a transverse intersection which is contained in $\alpha'_\epsilon \cap \beta'_\epsilon$. We call such a pair of segments $\alpha_\epsilon, \beta_\epsilon$ an *ϵ-trap* for the intersection p. Or, if ϵ is understood, just a *trap* for p.

Now let S be a closed hyperbolic surface, and let $\gamma \subset S$ be a closed geodesic. Suppose α, β project to γ by the covering projection.

Lemma 3.1. *Let α, β, γ, S be as above, and let $\alpha_\epsilon, \beta_\epsilon$ be an ϵ-trap for p, which projects to a self-intersection of γ. Suppose 8ϵ is less than the length of the shortest nontrivial curve on S. Then $\text{length}(\alpha_\epsilon) < \text{length}(\gamma) + 4\epsilon$ and similarly for β_ϵ.*

Proof. By the definition of an ϵ-trap, the endpoints of α_ϵ and β_ϵ are 2ϵ apart. By the triangle inequality, there is an estimate

$$|\text{length}(\alpha_\epsilon) - \text{length}(\beta_\epsilon)| < 4\epsilon$$

Suppose $\text{length}(\alpha_\epsilon) \geq \text{length}(\gamma) + 4\epsilon$ and therefore $\text{length}(\beta_\epsilon) \geq \text{length}(\gamma)$. By hypothesis there are elements $a, b \in \pi_1(S)$ in the conjugacy class of γ which stabilize α, β respectively and act as translations through a distance $\text{length}(\gamma)$. Let q be an endpoint of α_ϵ. Then $d(b^{-1}a(q), q) \leq 8\epsilon$. But by hypothesis, this is shorter than the length of the shortest nontrivial curve on S, so $b = a$ and $\beta = \alpha$, contrary to hypothesis. It follows that $\text{length}(\alpha_\epsilon) < \text{length}(\gamma) + 4\epsilon$ and the lemma is proved. □

Lemma 3.2. *Let S, ϵ be as above. Let γ be a closed geodesic on S with a transverse self-intersection. Let α be a geodesic in the hyperbolic plane covering γ, and let σ be a geodesic segment in the hyperbolic plane which is ϵ-close to a segment of α, and satisfies $\text{length}(\sigma) > 2 \cdot \text{length}(\gamma) + 4\epsilon$. Then the projection of σ to S has a transverse self-intersection.*

Proof. Let β be another geodesic in the hyperbolic plane covering γ such that $\alpha \cap \beta$ contains a point p projecting to a transverse self-intersection of γ, and let $\alpha_\epsilon, \beta_\epsilon$ be an ϵ-trap for p. By Lemma 3.1, $\text{length}(\alpha_\epsilon) + \text{length}(\gamma) \leq \text{length}(\sigma)$ so there is a translate of σ by an element of the deck group which contains a segment α'_ϵ which is ϵ-close to α_ϵ. Similarly, there is a translate of σ containing a segment β'_ϵ which is ϵ-close to β_ϵ. Since $\alpha_\epsilon, \beta_\epsilon$ is an ϵ-trap for p, the segments $\alpha'_\epsilon, \beta'_\epsilon$ intersect transversely. The projections of α'_ϵ and β'_ϵ are both contained in the projection of σ, and therefore this projection contains a point of self-intersection, as claimed. □

Remark 3.3. Chris Leininger has observed that for every non-simple primitive homotopy class of loop on a surface S, there is some hyperbolic structure on S for which the geodesic representative intersects itself at a definite angle. This observation, together with the fact that w_n is characteristic, could be used in place of the results in this section in the proof of Theorem A.

4. Proof of Theorem A

We now give the proof of Theorem A.

Proof. Fix a hyperbolic structure on S. Fix a generating set A for $\pi_1(S)$, and let δ be such that the Cayley graph with respect to this generating set is δ-hyperbolic. Let K, ϵ' be such that the Cayley graph $C_A(\pi_1(S))$ is (K, ϵ') quasi-isometric to the

hyperbolic plane (identified with the universal cover of S) by a fixed equivariant quasi-isometry. Note that K, ϵ' can be chosen to depend only on S.

Let $a \in \pi_1(S)$ have $\mathrm{cr}(a) > 0$. There is a constant $C_1(\delta, |A|)$ depending only on δ and the cardinality $|A|$ such that a^{C_1} has an axis in $C_A(\pi_1(S))$. We replace a by a suitable conjugate b of a^{C_1} whose axis contains id. Note that $\gamma(b)$ is just a C_1-multiple of $\gamma(a)$, and therefore contains $C_1^2\mathrm{cr}(a)$ transverse self-intersections. Since C_1 depends only on $\delta, |A|$, and therefore only on S, it suffices to prove the theorem with b in place of a.

Let $l \subset C_A(\pi_1(S))$ be the (oriented) axis of b, which by hypothesis passes through id. Let N be a sufficiently large number (we will say how large in a moment), and let σ be the oriented segment of l from id to b^N. Define the quasimorphism h_σ as in Definition 2.2.

Claim 1 of Theorem A′ in [5] says that if the translation length of b^N is sufficiently long (depending only on δ and $|A|$), then $c_{\sigma^{-1}}(b^{Nm}) = 0$ for all positive m. From the definition, $c_\sigma(b^{Nm}) = m$ for all positive m, and therefore we have an equality $h_\sigma(b^{Nm}) = m$, valid (by symmetry) for all integers m. We can make the translation length of b^N long enough by replacing b by a proper power if necessary, absorbing the constant (multiplicatively) into C_1 as above.

On the other hand, let $e \in S_n$. Observe that e^{C_1} has an axis l_e. Note that $\mathrm{cr}(e^{C_1}) \le {C_1}^2 n$ by the definition of S_n. How many copies of σ can there be in a realizing path for c_σ on e^{C_1}? Each copy of σ in such a realizing path defines a segment in l_e which is in the C_2-neighborhood of a translate of l, where C_2 is as in Lemma 2.4.

It follows that there is a constant C_3, depending only on C_2, K, ϵ', such that suitable geodesics $\tilde{\gamma}(e)$ and $\tilde{\gamma}(b)$ covering $\gamma(e), \gamma(b)$ respectively are distance at most C_3 apart on segments of length at least $N \cdot \mathrm{length}(\gamma(b))$.

By convexity of distance in hyperbolic space, for any $\epsilon > 0$ there is a constant $C_4(\epsilon)$ such that these lifts are ϵ-close on segments on length at least $N \cdot \mathrm{length}(\gamma(b)) - C_4$. This tells us how to choose N. We choose ϵ such that 8ϵ is less than the length of the shortest nontrivial loop on S, and choose N so that

$$N \cdot \mathrm{length}(\gamma(b)) - C_4 > 2 \cdot \mathrm{length}(\gamma(b)) + 4\epsilon$$

in order to be able to apply Lemma 3.2.

Note that $\mathrm{length}(\gamma(b)) \ge 8\epsilon$, and ϵ is bounded below by a positive constant depending only on the geometry of S, so N, ϵ as above depend only on S.

Let τ be a segment of $\tilde{\gamma}(e)$ coming from a copy of σ in a realizing path for c_σ on e^{C_1}, which is ϵ-close to a segment of $\tilde{\gamma}(b)$. Since τ by hypothesis has length at least $2 \cdot \mathrm{length}(\gamma(b)) + 4\epsilon$, by Lemma 3.2, the projection of τ to S contributes at least one transverse self-intersection to $\gamma(e^{C_1})$.

If a realizing path for e^{C_1} contains p disjoint copies of σ, we obtain p such segments $\tau_1, \ldots, \tau_p$, and therefore at least p^2 self-intersections of $\gamma(e^{C_1})$. Since $\mathrm{cr}(e^{C_1}) = C_1^2\mathrm{cr}(e) \le C_1^2 n$ we obtain an estimate $p^2 \le C_1^2 n$.

Obviously a similar estimate holds for copies of σ^{-1} in e^{C_1}. Therefore we obtain

$$|h_\sigma(e^{C_1})| \le p \le C_1\sqrt{n}$$

Note that after homogenizing, we will obtain a similar estimate for e.

By Lemma 2.5, the defect of h_σ depends only on S, so we homogenize h_σ and get a homogeneous quasimorphism $\overline{h}_\sigma$ with defect depending only on S, and satisfying

$$\overline{h}_\sigma(b^m) = \frac{m}{N}, \quad \overline{h}_\sigma(e) \le C_5\sqrt{n} + C_6$$

for any integer m and any $e \in S_n$, where N, C_5, C_6 depend only on S.

By the defining property of quasimorphisms (that they are additive with bounded error), we get a bound of the form

$$w_n(b^m) \ge \frac{C_7 m}{\sqrt{n} + C_8} - C_9$$

where all constants depend only on S. Since $w_n(a^{mC_1}) = w_n(b^m)$, and a is an arbitrary element with $\mathrm{cr}(a) > 0$, this proves Theorem A. □

Remark 4.1. In view of the estimates in this section, one might be tempted to guess that crossing number itself is an appropriate function with which to measure word length in $\pi_1(S)$. Nevertheless, one can construct examples of pairs of elements $a, b \in \pi_1(S)$ such that $\mathrm{cr}(a) = \mathrm{cr}(b) = 0$ but $\mathrm{cr}(ab)$ is as large as desired. For example, let $\gamma(b)$ be a very long simple geodesic which is very close (in the Hausdorff topology) to a minimal geodesic lamination, and let $\gamma(a)$ cross a collection of 10^{100} almost parallel strands of $\gamma(b)$. Let $p \in \gamma(a) \cap \gamma(b)$ be a basepoint, in order to pin down the based homotopy classes of a, b. Then the conjugacy class of ab contains a representative loop (not a geodesic) which consists of the geodesic $\gamma(a)$ followed by the geodesic $\gamma(b)$. Most of the self-intersections of the loop ab are essential, and therefore the geodesic $\gamma(ab)$ also has at least 10^{100} transverse self-intersections.

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